ACTIVITY CODE: 1903120021

B.Sc. 6th Semester (Honours) Examination, October 2020

Subject: Mathematics

Course ID: 62111 **Course Code: SH/MTH /601/C-13**

Course Title: Metric Spaces and Complex Analysis

Full Marks: 20 **Time: 1 Hour**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meanings.

1. **Answer *any two* of the following questions:** $(\mathbf{2 \times 2 = 4})$

a) Give an example of a function $f: \mathbb{C} \to \mathbb{C}$ which is continuous everywhere but nowhere differentiable.

b) Define contour in complex plane.

c) Suppose C is the unit circle centered at origin. Then find the value of the integration

$$\int_C \frac{e^z}{(z-2)} dz.$$

d) Define a complex entire function.

e) Give an example of a complete metric space and an incomplete metric space on the interval $(0,1) \subset \mathbb{R}$.

f) Show that the space $\mathbb{Q}$ is disconnected with respect to the usual metric.

g) Let $f: [0,1] \to [0,1]$ be defined by $f(x) = 0$ for $x \in \left[0, \frac{1}{2}\right]$ and $f(x) = \frac{1}{2}$ for $x \in \left(\frac{1}{2}, 1\right]$. Check whether f is a contraction mapping under usual metric.

h) State Banach fixed point theorem.

2. **Answer *any two* of the following questions:** $(\mathbf{5 \times 2 = 10})$

a) (i) Suppose $f(x+iy) = u(x,y) + 2i$ is an entire function. Show that u is constant.

(ii) Suppose f is an entire function satisfying the condition $|f(z)| \leq \frac{z^4}{\ln|z|}$ for all $|z| > 1$. Show that $f(z)$ is a polynomial of degree atmost 3. (2+3)

b) State and prove Cauchy's integral theorem.

c) (i) State Laurent's theorem.

(ii) Find the Laurent series of $\frac{1}{z^2+1}$ in the deleted neighbourhood of $z = i$. (2+3)

d) (i) Let $\{x_n\}$ be a Cauchy sequence in a metric space having a convergent subsequence. Show that $\{x_n\}$ is convergent.

(ii) Show that the space of all polynomials equipped with sup-metric is not complete. (2+3)

e) Prove that every sequentially compact metric space is compact.

f) Show that continuous image of a compact set is compact.

3. Answer *any two* from either a) or b) : **$(3 \times 2 = 6)$**

a) (i) Show that a compact metric space is totally bounded.

(ii) For any set A in a metric space, show that $diam\ A = diam\overline{A}$, $\overline{A}$ denotes the closure of A.

(iii) Give an example (with justification) of non-isometric homeomorphism.

(iv) Let f be a piecewise continuous complex function on a contour C of length L. If $|f(z)| \leq M$ for all z on C (M is a non-negative constant), then show that $\left|\int_C f(z)dz\right| \leq ML$.

b) (i) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{z^{3n}}{(3+i)^n} .$$

(ii) Show that for any polynomial p, $p(\mathbb{C}) = \mathbb{C}$. Does the converse hold? Support your answer.

(iii) Let $f(z) = u(x, y) + iv(x, y)$ be defined in a domain D such that u and v have continuous partial derivatives that satisfy the Cauchy-Riemann equations for all points in D. Then show that $f(z)$ is analytic in D.

(iv) Evaluate the integral $\int_C \frac{1}{(z^2+9)^2} dz$, where $C: |z - 2i| = 2$ in positive sense.

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